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On algebraicity of special values of L-functions for $\mathrm{SO}(\mathbf{V}) \times \mathrm{GL}_2$

By

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Abstract

In this paper, we announce the result of the joint work [8] with Masaaki Furusawa, in which we state a conjecture (see Conjecture 2) on the algebraicity of special values of tensor product L-functions for automorphic representations of $\mathrm{SO}(\mathbf{V})$ and GL_2 explicating Deligne periods and we show this conjecture at various critical points under certain assumption on the weight (see Theorem 3.1). This algebraicity result is a generalization of our previous result [7, Theorem 1] with respect to critical points and infinity types of automorphic representations of $\mathrm{SO}(\mathbf{V})$.

§ 1. Notation

Let $\mathbf{V}$ be a quadratic space over $\mathbb{Q}$ such that $\dim \mathbf{V} = n \geq 2$ and $\mathbf{V} \otimes_{\mathbb{Q}} \mathbb{R}$ is positive definite. When n is even, let $\chi_{\mathbf{V}}$ denote the quadratic character of $\mathbb{A}_{\mathbb{Q}}^{\times}$ given by

$$\chi_{\mathbf{V}}(x) = \left(x, (-1)^{n/2}\, d(\mathbf{V})\right)_{\mathbb{Q}}$$

where $(\ ,\ )_{\mathbb{Q}}$ is the Hilbert symbol of $\mathbb{Q}$ and $d(\mathbf{V})$ is the discriminant of $\mathbf{V}$. When n is odd, let $\chi_{\mathbf{V}}$ be the trivial character of $\mathbb{A}_{\mathbb{Q}}^{\times}$.

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We recall that by the highest weight theory, isomorphism classes of finite dimensional irreducible complex representations of $\mathrm{SO}(\mathbf{V},\mathbb{R})$ are completely parametrized by

$$\Lambda_n = \begin{cases} \left\{(m_1,\dots,m_{n'}) \in \mathbb{Z}^{n'} \mid m_1 \geq \cdots \geq m_{n'} \geq 0\right\} & \text{when } n \text{ is odd,} \\ \left\{(m_1,\dots,m_{n'}) \in \mathbb{Z}^{n'} \mid m_1 \geq \cdots \geq m_{n'-1} \geq |m_{n'}|\right\} & \text{when } n \text{ is even,} \end{cases}$$

with $n' = \left[\frac{n}{2}\right]$ (for example, see [12, Section 3]). Here $[x]$ denotes the integer such that $[x] \leq x < [x]+1$ for $x \in \mathbb{R}$.

For a Hecke character η of $\mathbb{A}_{\mathbb{Q}}^{\times}$ of finite order, let η_0 be its associated Dirichlet character. Then we denote by $\mathfrak{g}\,(\eta)$ the Gauss sum $\mathfrak{g}\,(\eta_0)$ for η_0.

For an integer $k \geq 2$, let $S_k^{\mathrm{new}}\,(\Gamma_0(N),\varepsilon)$ denote the set of normalized newforms for $\Gamma_0(N)$ of weight k with Nebentypus ε. For $f \in S_k^{\mathrm{new}}\,(\Gamma_0(N),\varepsilon)$, let

$$f(z) = \sum_{n=1}^{\infty} c_n(f) e^{2\pi i n z}$$

be the Fourier expansion of f at the infinity. Then for $\sigma \in \mathrm{Aut}(\mathbb{C})$, the group of field automorphisms of $\mathbb{C}$, we define $f^{\sigma} \in S_k^{\mathrm{new}}\,(\Gamma_0(N),\varepsilon^{\sigma})$ by

$$f^{\sigma}(z) = \sum_{n=1}^{\infty} c_n(f)^{\sigma} \cdot e^{2\pi i n z}.$$

For $\sigma \in \mathrm{Aut}(\mathbb{C})$, let us define the σ-twist of a complex representation (π, V) as in Waldspurger [22, 1.1]. Let V' be a $\mathbb{C}$-vector space with σ-linear (i.e. $t'(av) = \sigma(a)\, t'(v)$) isomorphism $t' : V \to V'$. Then we define the σ-twist of π by

$$\pi^{\sigma} := t' \circ \pi \circ t'^{-1}.$$

This definition is independent of t' and V' up to equivalence of representations.

Finally, for an automorphic representation π and $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define the σ-twist of π by

$$\pi^{\sigma} := \pi_{\mathrm{fin}}^{\sigma} \otimes \pi_{\infty}$$

where π_{fin} and π_{∞} are the finite and infinite part of π, respectively, and $\pi_{\mathrm{fin}}^{\sigma}$ is the σ-twist of π_{fin} as a complex representation.

§ 2. Deligne's conjecture and its explication for $\mathrm{SO}(\mathbf{V}) \times \mathrm{GL}_2$

In this section, we shall explicate Deligne's conjecture on the algebraicity of special values of motivic L-functions at critical points when a motive is given by a tensor product of motives corresponding to automorphic representations of $\mathrm{SO}(\mathbf{V})$ and GL_2.

Let us briefly recall Deligne's conjecture. Readers are referred to Deligne [6, Section 2] and Yoshida [23, Section 2]. In fact, we follow the latter rather closely.

Recall that there are various categories of pure motives. However it is not very important to consider a particular category since our purpose is not to discuss Deligne's conjecture itself but to give a conjecture on special values of L-functions for automorphic representations explicating Deligne periods. That is why we do not specify a category of motives, and we call an object of any category of motives merely motive.

Let $\mathcal{M}$ be a motive over $\mathbb{Q}$ with coefficients in a number field E. Let J_E denote the set of all embeddings of E into $\mathbb{C}$. For a finite place λ of E, let $H_\lambda(\mathcal{M})$ be the λ-adic realization of $\mathcal{M}$, which determines a λ-adic representation $\sigma_\lambda : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}\left(H_\lambda\left(\mathcal{M}\right)\right)$. For a prime number p such that $(\lambda, p) = 1$, put

$$Z_p(\mathcal{M}, X) = \det\left(1 - X\,\sigma_\lambda\left(\Phi_p\right)|_{H_\lambda(\mathcal{M})^{I_p}}\right)^{-1},$$

where Φ_p denotes a geometric Frobenius at p and I_p denotes the inertia group at p. Then there exists a finite set S of primes, so that σ_λ is unramifed at a prime number $p \notin S$ and $Z_p\left(\mathcal{M}, X\right)^{-1} \in E_\lambda[X]$ is independent of λ. In general, it is conjectured that this local factor is independent of λ for any p and that we have $Z_p\left(\mathcal{M}, X\right)^{-1} \in E[X]$ (see [6, 1.1, (1.2.1)]). From now on, we assume these conjectures. Then for $\sigma \in J_E$, put

$$L_p(\sigma, \mathcal{M}, s) = \sigma Z_p\left(\mathcal{M}, p^{-s}\right),$$

and

$$L(\sigma, \mathcal{M}, s) = \prod_p L_p(\sigma, \mathcal{M}, s).$$

Let

$$H_B(\mathcal{M}) = \bigoplus_{p,q\in\mathbb{Z}} H^{p\,q}(\mathcal{M})$$

be the Hodge decomposition of the Betti realization $H_B(\mathcal{M})$ of $\mathcal{M}$. Suppose that $\mathcal{M}$ is of pure weight w, i.e., $H^{p\,q}(\mathcal{M}) = 0$ whenever $p + q \neq w$. Then it is conjectured that $L(\sigma, \mathcal{M}, s)$ converges absolutely for $\mathrm{Re}(s) > \frac{w}{2} + 1$ and has a meromorphic continuation to the whole plane $\mathbb{C}$. Further, associated to the Hodge structure of the Betti realization $H_B(\mathcal{M})$, the archimedean local L-factor $L_\infty(\mathcal{M}, s)$ is defined independently from $\sigma \in J_E$ (see Serre [19, Section 3]). Then it is conjectured that $L(\sigma, \mathcal{M}, s)L_\infty(\mathcal{M}, s)$ has a functional equation with respect to $s \mapsto w + 1 - s$. Hereafter, we assume these conjectures.

We say that an integer $m \in \mathbb{Z}$ is critical for $\mathcal{M}$ if neither $L_\infty(s, \mathcal{M})$ nor $L_\infty(w + 1 - s, \mathcal{M})$ has a pole at $s = m$. If m is critical for $\mathcal{M}$, then we should have

$$p < m \leq q \quad \text{whenever } H^{p\,q}(\mathcal{M}) \neq \{0\},\ p < q. \tag{2.1}$$

The condition (2.1) is sufficient for m to be critical if w is odd. On the other hand, if w is even and we put $w = 2p$, the complex conjugation c_∞ must act on $H^{p\,p}(\mathcal{M})$ by the scalar for which $\mathcal{M}$ has a critical integer. Put

$$c_\infty = (-1)^{p+\varepsilon}, \quad \varepsilon = 0 \text{ or } 1 \text{ on } H^{p\,p}(\mathcal{M}).$$

Then we should have

$$\begin{cases} m > p - \varepsilon & \text{if } p + \varepsilon + m \text{ is even,} \\ m < p + \varepsilon + 1 & \text{if } p + \varepsilon + m \text{ is odd.} \end{cases} \tag{2.2}$$

Moreover (2.1) and (2.2) are sufficient for $m \in \mathbb{Z}$ to be critical for $\mathcal{M}$. Assume that $\mathcal{M}$ has a critical integer. Then we define Deligne periods $c^\pm(\mathcal{M})$ (see [6, p.323]) as follows.

Let $\{F^i(H_{DR}(\mathcal{M}))\}$ be the Hodge filtration of $H_{DR}(\mathcal{M})$. When w is odd, let $F^+(\mathcal{M}) = F^-(\mathcal{M}) = F^{(w-1)/2}(\mathcal{M})$. When $w = 2p$ is even and $p + \varepsilon$ is even, let

$$F^+(\mathcal{M}) = F^p(H_{DR}(\mathcal{M})) \quad \text{and} \quad F^-(\mathcal{M}) = F^{p+1}(H_{DR}(\mathcal{M})).$$

When $w = 2p$ is even and $p + \varepsilon$ is odd, let

$$F^+(\mathcal{M}) = F^{p+1}(H_{DR}(\mathcal{M})) \quad \text{and} \quad F^-(\mathcal{M}) = F^p(H_{DR}(\mathcal{M})).$$

Let us put

$$H^+_{DR}(\mathcal{M}) = H_{DR}(\mathcal{M})/F^-(\mathcal{M}) \quad \text{and} \quad H^-_{DR}(\mathcal{M}) = H_{DR}(\mathcal{M})/F^+(\mathcal{M}).$$

Then we have the canonical isomorphisms

$$I^\pm : H^\pm_B(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^\pm_{DR}(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{C}$$

as $E \otimes_{\mathbb{Q}} \mathbb{C}$-modules. Let

$$c^\pm(\mathcal{M}) = \det(I^\pm) \in (E \otimes_{\mathbb{Q}} \mathbb{C})^\times$$

be the determinants calculated by an E-rational basis. Then Deligne periods $c^\pm(\mathcal{M})$ are determined up to a multiplication by elements of $E^\times$. Further, we remark that Deligne periods are written as periods integrals (see [6, 1.7]).

Now, let us state Deligne's conjecture. We define a function $L^*(\mathcal{M}, s)$ taking values in $E \otimes_{\mathbb{Q}} \mathbb{C}$ by assigning $\{L(\sigma, \mathcal{M}, s)\}_{\sigma \in J_E}$ through the identification $E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}^{J_E}$. Then Deligne conjectured that (see [6, Conjecture 2.8])

$$\frac{L^*(\mathcal{M}, m)}{(1 \otimes (2\pi\sqrt{-1})^{d^\pm(\mathcal{M})m})c^\pm(\mathcal{M})} \in E \tag{2.3}$$

where double-sign corresponds. Here $d^\pm(\mathcal{M})$ is the dimension of $\pm$-eigen space of the Betti realization of $\mathcal{M}$, and E is embedded into $\mathbb{C}^{J_E}$ by $e \mapsto (\sigma(e))_\sigma$.

Our aim is to explicate Deligne periods for a tensor product motive $M \otimes N$ when motives M and N correspond to automorphic representations of $\mathrm{SO}(\mathbf{V}, \mathbb{A}_{\mathbb{Q}})$ and $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, respectively. Recall that in various situations, Deligne periods for tensor product motives are considered (for example, see Bhagwat–Raghuram [2], Blasius [3] and Yoshida [23], [24]). We shall consider Deligne periods in the following situation.

Suppose that $f \in S_k^{\mathrm{new}}(\Gamma_0(N), \varepsilon)$ is a Hecke eigenform. We denote by $\pi = \pi(f)$ the irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to f. Let τ be an irreducible automorphic representation of $\mathrm{SO}(\mathbf{V}, \mathbb{A}_{\mathbb{Q}})$ whose infinite component τ_∞ is an irreducible representation of $\mathrm{SO}(\mathbf{V}, \mathbb{R})$ with the highest weight $(m_1, \cdots, m_{n'}) \in \Lambda_n$. For our computation, we impose three assumptions. First, we assume that a special case of the conjecture by Clozel [5, Conjecture 4.5] holds, which states an existence of motives corresponding to algebraic automorphic representations of general linear groups. Indeed, we assume the following conjecture.

Conjecture 1 (Conjecture 4.5 in Clozel [5]). *Let Π be an irreducible algebraic automorphic representation of $\mathrm{GL}_{2r}(\mathbb{A}_{\mathbb{Q}})$. Then there exists a motive M_Π over $\mathbb{Q}$ with coefficients in a number field E_Π such that M_Π is of pure weight and we have*

$$L_v(\sigma, M_\Pi, s) = L\left(s - \frac{1}{2}, \Pi_v^\sigma\right)$$

for any place v of $\mathbb{Q}$ and $\sigma \in J_{E_\Pi}$.

Second of all, we impose the following two assumptions on τ.

Assumptions.

1. There exists an irreducible automorphic representation Π_τ of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ such that $(\Pi_\tau)_v$ is the local transfer of τ_v for almost all finite places and the infinite place v of $\mathbb{Q}$ where
$$m = \begin{cases} n-1 & \text{when } n \text{ is odd,} \\ n & \text{when } n \text{ is even,} \end{cases}$$
with $n = \dim \mathbf{V}$.

2. τ is tempered.

We give some remarks on these assumptions. In Arthur [1], he showed an existence of a functorial lift from quasi-split inner forms of $\mathrm{SO}(\mathbf{V})$ to GL_m, conditional on the stabilization of twisted trace formulas. Further, when τ is tempered (i.e. the second assumption holds), he states an existence of the functorial lift in the first assumption without a proof (see Arthur [1, Theorem 9.5.3]). One can be optimistic that the first assumption for our τ will be verified in a near future.

Further, we would like to give a remark on an algebraicity of automorphic representations of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ given in the first assumption. In order to simplify our argument in this remark, we suppose that $\dim \mathbf{V}$ is even. However, as we remark later, a similar result holds even when $\dim \mathbf{V}$ is odd.

Let us recall the Langlands parameter of τ_∞. Let $W_\mathbb{R}$ be the Weil group of $\mathbb{R}$, which contains $\mathbb{C}^\times$ as a normal subgroup of index two. For each $a \in \frac{1}{2}\mathbb{Z}$, we define a unitary character $\chi_a : \mathbb{C}^\times \to \mathbb{C}^1$ by

$$\chi_a(z) = \left(\frac{z}{\bar{z}}\right)^a$$

and we define a two-dimensional representation of $W_\mathbb{R}$ by

$$V(a) = \mathrm{Ind}_{\mathbb{C}^\times}^{W_\mathbb{R}}(\chi_a).$$

We know that the Langlands parameter of τ_∞ is given by

$$\bigoplus_{i=1}^{n/2} V(m_i + n - i)$$

where $(m_1, \ldots, m_{n/2})$ is the highest weight of τ_∞. Then we can easily show that $(\Pi_\tau)_\infty$ is algebraic in the sense of Clozel [5]. Since Π_τ should be an isobaric automorphic representation from the second assumption, Π_τ is an algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. We also note that Π_τ is uniquely determined by the strong multiplicity one theorem for isobaric automorphic representations. On the other hand, when $\dim \mathbf{V}$ is odd, in a similar argument, we see that $\Pi_\tau \otimes |-|^{1/2}$ is an algebraic automorphic representation.

Let us go back to a situation with no assumption on $\dim \mathbf{V}$.

Then the above remark and Conjecture 1 imply that there exists a motive M_τ over $\mathbb{Q}$ with coefficients in a number field E_τ such that M_τ is of pure weight and we have

$$L_v(\sigma, M_\tau, s) = L\left(s - \frac{i}{2}, \tau_v^\sigma\right) \quad \textit{where } i = \begin{cases} 1 & \textit{when } n \textit{ is odd,} \\ 0 & \textit{when } n \textit{ is even,} \end{cases}$$

for any place v of $\mathbb{Q}$ and $\sigma \in J_{E_\tau}$. Here we define $L(s, \tau_v) = L(s, (\Pi_\tau)_v)$ for any place v of $\mathbb{Q}$.

On the other hand, Scholl [18, Theorem 1.2.4] showed that there exists a rank two (Grothendieck) motive N_π over $\mathbb{Q}$ with coefficients in a number field $\mathbb{Q}(f)$ such that N_π is of pure weight and we have

$$L_v(\sigma, N_\pi, s) = L\left(s - \frac{j}{2}, \pi_v^\sigma\right) \quad \text{where } j = \begin{cases} 1 & \text{when } k \text{ is even,} \\ 0 & \text{when } k \text{ is odd,} \end{cases}$$

for any place v of $\mathbb{Q}$ and $\sigma \in J_{\mathbb{Q}(f)}$. Here, $\mathbb{Q}(f)$ is the algebraic number field generated by Fourier coefficients of f over $\mathbb{Q}$. Then we computed $c^{\pm}(M_\tau \otimes N_\pi)$ in Appendix to [8], and we obtained the following formulas.

Lemma 2.1 (Appendix to [8]). *Suppose all assumptions above, and assume that*

$$k - 2m_1 \geq n.$$

Let E be the composite field of $\mathbb{Q}(f)$ and E_τ. Then we have the following formulas as elements of $(E \otimes \mathbb{C})^\times / E^\times$ (Recall that Deligne periods $c^{\pm}(M_\tau \otimes N_\pi)$ are determined up to a multiplication by elements of $E^\times$). When n is odd, we have

$$c^{\pm}(M_\tau \otimes N_\pi) = \begin{cases} \left\{ \left(\left(2\pi\sqrt{-1}\right)^{-2} \cdot J(f^\sigma) \right)^{\left[\frac{n}{2}\right]} \right\}_{\sigma \in J_E} & \text{if } k \text{ is even,} \\ \left\{ \left(\left(2\pi\sqrt{-1}\right)^{-1} \cdot J(f^\sigma) \right)^{\left[\frac{n}{2}\right]} \right\}_{\sigma \in J_E} & \text{if } k \text{ is odd.} \end{cases}$$

When n is even, we have

$$c^{\pm}(M_\tau \otimes N_\pi) = \begin{cases} \mathfrak{g}(\chi_V) \left\{ \left(\left(2\pi\sqrt{-1}\right)^{-1} \cdot J(f^\sigma) \right)^{\left[\frac{n}{2}\right]} \right\}_{\sigma \in J_E} & \text{if } k \text{ is even,} \\ \mathfrak{g}(\chi_V) \left\{ \left(J(f^\sigma) \right)^{\left[\frac{n}{2}\right]} \right\}_{\sigma \in J_E} & \text{if } k \text{ is odd.} \end{cases}$$

Here $J(f)$ is defined by

$$J(f) = \pi^k \, \mathfrak{g}(\varepsilon) \, \langle f, f \rangle$$

where

$$\langle f, f \rangle = \mathrm{vol}\left(\Gamma_0(N)\backslash\mathfrak{h}\right)^{-1} \int_{\Gamma_0(N)\backslash\mathfrak{h}} |f(z)|^2 y^{k-2} \, dx \, dy$$

with $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

Because of this lemma, we may explicate Deligne's conjecture (2.3) and we may state the following conjecture on the algebraicity of special values of L-functions for $\mathrm{SO}(\mathbf{V}) \times \mathrm{GL}_2$.

Conjecture 2 (Appendix to [8]). *Let $\pi(f)$ (resp. τ) be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ (resp. $\mathrm{SO}(\mathbf{V}, \mathbb{A}_\mathbb{Q})$) as above. Let us define the field of rationality of τ by*

$$\mathbb{Q}(\tau) = \{a \in \mathbb{C} \mid \sigma(a) = a \text{ for any } \sigma \in \mathrm{Aut}(\mathbb{C}) \text{ such that } \tau_{\mathrm{fin}}^\sigma \simeq \tau_{\mathrm{fin}}\},$$

and we write the composite field of $\mathbb{Q}(\tau)$ *and* $\mathbb{Q}(f)$ *by* $\mathbb{Q}(f,\tau)$*. Suppose that we have*

$$k - 2m_1 \geq n.$$

Then for m *such that*

$$m = \frac{k-2m_1-n+1}{2} - l \quad \text{where } l \in \mathbb{Z} \text{ and } 0 \leq l \leq \frac{k-2m_1-n}{2},$$

we have

$$\frac{L\left(m, \pi(f) \otimes \tau\right)}{\left(2\pi\sqrt{-1}\right)^{2m\cdot\left[\frac{n}{2}\right]} \mathfrak{g}\left(\chi_{\mathbf{V}}\right) J\left(f\right)^{\left[\frac{n}{2}\right]}} \in \mathbb{Q}(f,\tau)$$

where the L*-function* $L\left(s, \pi\left(f\right) \otimes \tau\right)$ *is normalized so that it has the functional equation with respect to* $s \mapsto 1-s$*.*

Remark 1. When $n=2$, we have

$$\mathrm{SO}(\mathbf{V}) \simeq E^1 = \left\{a \in E \mid N_{E/F}(a) = 1\right\}$$

for some imaginary quadratic extension E of $\mathbb{Q}$. Then our L-function is a Rankin–Selberg L-function for $\mathrm{GL}_2 \times \mathrm{GL}_2$. Moreover, when $n=3$, we have

$$\mathrm{SO}(\mathbf{V}) \simeq D^\times / \mathbb{Q}^\times$$

for some definite quaternion division algebra over $\mathbb{Q}$. As in the previous case, by the Jacquet–Langlands correspondence, our L-function is regarded as a Rankin–Selberg L-function for $\mathrm{GL}_2 \times \mathrm{GL}_2$. Hence, for these cases, our conjecture is indeed a theorem of Shimura [20, Theorem 3].

Remark 2. Suppose $n=4$. Since we have

$$\mathrm{SO}\left(D\right) \simeq \left\{(d_1, d_2) \in D^\times \times D^\times \mid n_D\left(d_1\right) = n_D\left(d_2\right)\right\} / \left\{(a,a) \mid a \in \mathbb{Q}^\times\right\}$$

where D is a definite quaternion division algebra over $\mathbb{Q}$ and n_D denotes its reduced norm, our L-function may be regarded as a Rankin triple product L-function for GL_2 because of the Jacquet-Langlands correspondence. Note that Deligne's conjecture for the Rankin triple L-function was explicated by Blasius in [3]. Then our conjecture corresponds to the *unbalanced* case in the following conjecture.

Conjecture 3 (Blasius [3]). *For* $i=1,2,3$*, let* $f_i \in S_{k_i}^{\mathrm{new}}\left(\Gamma_0\left(N_i\right), \varepsilon_i\right)$ *such that* $k_1 \geq k_2 \geq k_3$*. Let* $\mathbb{Q}\left(f_1, f_2, f_3\right)$ *be the number field generated by the Fourier coefficients of* f_1*,* f_2 *and* f_3 *over* $\mathbb{Q}$*. We denote by* $L\left(s, f_1 \otimes f_2 \otimes f_3\right)$ *the Rankin triple product* L*-function. Here we normalize the* L*-function so that it has a functional equation with respect to* $s \mapsto k_1 + k_2 + k_3 - 2 - s$*.*

1. *(Balanced case) Suppose that $k_1 < k_2 + k_3$. Let $A = 3 - k_1 - k_2 - k_3$. Then for $n \in \mathbb{Z}$ such that $k_1 \leq n \leq k_2 + k_3 - 2$, we have*
$$\frac{L(n, f_1 \otimes f_2 \otimes f_3)}{\pi^{4n+A}\, \mathfrak{g}(\varepsilon_1\varepsilon_2\varepsilon_3)^2 \langle f_1, f_1\rangle \langle f_2, f_2\rangle \langle f_3, f_3\rangle} \in \mathbb{Q}(f_1, f_2, f_3).$$

2. *(Unbalanced case) Suppose that $k_1 \geq k_2 + k_3$. Let $B = 4 - 2k_2 - 2k_3$. Then for $n \in \mathbb{Z}$ such that $k_2 + k_3 - 1 \leq n \leq k_1 - 1$, we have*
$$\frac{L(n, f_1 \otimes f_2 \otimes f_3)}{\pi^{4n+B}\, \mathfrak{g}(\varepsilon_1\varepsilon_2\varepsilon_3)^2 \langle f_1, f_1\rangle^2} \in \mathbb{Q}(f_1, f_2, f_3).$$

In the balanced case, using a remarkable integral representation by Garrett [9], algebraicity of the special values of Rankin triple product L-functions have been studied by Garrett [9], Orloff [16], Satoh [17], Harris–Kudla [14], Garrett–Harris [10] and Böcherer–Schulze-Pillot [4]. On the other hand, Harris–Kudla [14] showed Jacquet's conjecture, and they proved Blasius' conjecture at the central critical point not only in the balanced case but also in the unbalanced case under the assumption $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$.

§ 3. Main Result

We now state our main result in [8].

Theorem 3.1. *Let $\pi(f)$ and τ be as in Conjecture 2. Suppose that*

$$k - 2m_1 > 2n. \tag{3.1}$$

Then there exists a finite set S° of places of $\mathbb{Q}$ containing the infinite place such that

$$P_S(m, f, \tau) := \frac{L_S(m, \pi(f) \otimes \tau)}{\left(2\pi\sqrt{-1}\right)^{2m\cdot\left[\frac{n}{2}\right]} \mathfrak{g}(\chi_{\mathbf{V}})\, J(f)^{\left[\frac{n}{2}\right]}} \in \overline{\mathbb{Q}} \tag{3.2}$$

and

$$P_S(m, f, \tau)^\sigma = P_S(m, f^\sigma, \tau^\sigma) \quad \text{for any } \sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$$

holds for any finite set S of places of F containing S° and for m such that

$$m = \frac{k - 2m_1 - n + 1}{2} - l \quad \text{where } l \in \mathbb{Z} \text{ and } 0 \leq l \leq \frac{k - 2m_1 - 2n - 1}{2}. \tag{3.3}$$

In particular,
$$P_S(m, f, \tau) \in \mathbb{Q}(f, \tau).$$

Here $L_S(s, \pi(f) \otimes \tau)$ denotes the partial L-function defined by
$$L_S(s, \pi(f) \otimes \tau) = \prod_{v \notin S} L(s, \pi(f)_v \otimes \tau_v).$$

Remark 3. As we shall discuss below, our proof relies on the algebraicity of Fourier coefficients of Eisenstein series on Shimura varieties in the domain of absolute convergence proved by Harris [13, Theorem 8.5]. Thus the upper bound for l in (3.3) is forced upon us and the condition (3.1) becomes necessary so that the set of critical points m satisfying (3.3) to be non-empty.

Remark 4. In our paper [8], Conjecture 2 and Theorem 3.1 are, indeed, stated and proved, respectively, over arbitrary totally real number fields. The setting for this note is over a rational number field $\mathbb{Q}$ just to simplify the exposition.

Remark 5. In our previous paper [7], we showed only the algebraicity (3.2) in the simplest case, namely when τ_∞ is trivial and m is the rightmost critical point. Thus Theorem 3.1 is a generalization of [7, Theorem 1].

Remark 6. If we have the local Langlands correspondence for $\mathrm{SO}\,(\mathbf{V}, \mathbb{Q}_p)$ at each prime p, then by an argument similar to the one given in [7, Section 6], the algberaicity for complete L-functions holds (see the proof of Corollary 3.3 when $\dim \mathbf{V} = 4$).

Let us give an outline of our proof of Theorem 3.1. For simplicity, we merely prove algebraicity of special values of our L-function.

Outline of the proof of Theorem 3.1. In order to study our L-function, we use an integral representation by Ginzburg–Piatetski-Shapiro–Rallis [11], which is given as follows.

We choose a co-dimension one subspace $\mathbf{W}$ of $\mathbf{V}$, and let $\mathbb{V} = \mathbf{W} \oplus \mathbb{H} \oplus \mathbb{H}$ where $\mathbb{H}$ is the hyperbolic plane. We note that the symmetric space associated to $\mathrm{SO}(\mathbb{V})$ gives a type IV tube domain.

We know that $\mathrm{SO}(\mathbb{V})$ has a maximal parabolic subgroup P whose Levi component is $\mathrm{GL}_2 \times \mathrm{SO}(W)$. Moreover, $\mathrm{SO}(\mathbb{V})$ has another maximal parabolic subgroup Q whose unipotent radical U_Q is abelian.

Let us take an irreducible automorphic representation (ρ, V_ρ) of $\mathrm{SO}(\mathbf{W}, \mathbb{A}_\mathbb{Q})$ such that a $\mathrm{SO}(\mathbf{W}, \mathbb{A}_\mathbb{Q})$-invariant bilinear form on $V_\tau \times V_\rho$ given by

$$V_\tau \times V_\rho \ni (\phi_\tau, \phi_\rho) \mapsto \int_{\mathrm{SO}(\mathbf{W},\mathbb{Q})\backslash \mathrm{SO}(\mathbf{W},\mathbb{A}_\mathbb{Q})} \phi_\tau(g)\phi_\rho(g)\, dg$$

is not identically zero. For $f_s \in \mathrm{Ind}_{P(\mathbb{A}_\mathbb{Q})}^{\mathrm{SO}(\mathbb{V},\mathbb{A}_\mathbb{Q})}(\pi \otimes \rho \otimes \delta_P^s)$, we define an Eisenstein series by

$$E(g, s) := \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{SO}(\mathbb{V},\mathbb{Q})} f_s(\gamma g).$$

Let us take a character θ of $U_Q(\mathbb{A}_\mathbb{Q})/U_Q(\mathbb{Q})$ such that the identity component of the stabilizer of θ in the Levi component $M_Q(\mathbb{A}_\mathbb{Q})$ of $Q(\mathbb{A}_\mathbb{Q})$ is $\mathrm{SO}(\mathbf{V}, \mathbb{A}_\mathbb{Q})$. Then the global zeta integral is defined by

$$Z(s) = \int_{(\mathrm{SO}(\mathbf{V}) \times U_Q)(\mathbb{Q}) \backslash (\mathrm{SO}(\mathbf{V}) \times U_Q)(\mathbb{A}_\mathbb{Q})} E(h,s)\Phi(g)\theta(u)\, dh\, du$$

for $\Phi \in V_\tau$ and $f_s \in \mathrm{Ind}_{P(\mathbb{A}_\mathbb{Q})}^{\mathrm{SO}(\mathbb{V},\mathbb{A}_\mathbb{Q})}(\pi \otimes \rho \otimes \delta_P^s)$. When Φ and f_s are decomposable, this global integral becomes an infinite product of local integrals

$$Z(s) = \prod_v Z_v(s),$$

and Ginzburg, Piatetski-Shapiro and Rallis computed local zeta integrals explicitly when all the data involved are unramified.

Theorem 3.2 (Ginzburg–Piatetski-Shapiro–Rallis [11]). *Let v be a finite place of $\mathbb{Q}$ satisfying the following conditions.*

1. $\mathrm{SO}(\mathbf{V}, \mathbb{Q}_v)$ *and* $\mathrm{SO}(\mathbf{W}, \mathbb{Q}_v)$ *are quasi-split.*
2. π_v, ρ_v, τ_v *and* θ_v *are unramified,*
3. v *does not lie over 2,*

Suppose that local components of Φ and f_s at the place v are unramified and suitably normalized. For $\mathrm{Re}(s) >> 0$, *we have*

$$Z_v(s) = \frac{L\left(ns + \frac{1}{2}, \pi_v \otimes \tau_v\right)}{L\left(ns + 1, \pi_v \otimes \rho_v\right) L\left(2ns, \pi_v, r\right)}$$

where $n = \dim\ \mathbf{V}$ *and*

$$r = \begin{cases} \wedge^2 & \textit{when } n \textit{ is odd,} \\ \mathrm{Sym}^2 & \textit{when } n \textit{ is even.} \end{cases}$$

For an integer or a half of integer m given in (3.3), we let $s_m = \frac{1}{n}\left(m - \frac{1}{2}\right)$. Then we may choose Φ and f_s satisfying following three conditions.

1. At $s = s_m$, the local zeta integral $Z_v(s)$ converges and becomes a non-zero algebraic number.
2. At $s = s_m$, the Eisenstein series absolutely converges and becomes holomorphic Eisenstein series on the type IV tube domain. Further, it has algebraic Fourier coefficients by Harris [13, Theorem 8.5].
3. The global zeta integral $Z(s_m)$ becomes algebraic number (since $Z(s_m)$ can be written as a finite sum of Fourier coefficients of the holomorphic Eisenstein series).

With the above choice, we may compute $Z_\infty(s_m)$ explicitly, and we get the following identity

$$Z(s_m) = \frac{P_S(m, f, \tau)}{P_S\left(m + \frac{1}{2}, f, \rho\right) Q_S(2m, \pi, r)} \cdot \prod_{v \in S \setminus \{\infty\}} Z_v(s_m)$$

for some finite set of places S of $\mathbb{Q}$ containing the archimedean place ∞. Here, we put

$$Q_S(2m, \pi, r) = \begin{cases} \dfrac{L_S\left(2m, \pi, \wedge^2\right)}{\left(2\pi\sqrt{-1}\right)^{2m} \mathfrak{g}(\varepsilon)} & \text{when } n \text{ is odd,} \\[2ex] \dfrac{L_S\left(2m, \pi, \mathrm{Sym}^2\right)}{\left(2\pi\sqrt{-1}\right)^{4m} \mathfrak{g}(\varepsilon) J(f)} & \text{when } n \text{ is even.} \end{cases}$$

We know that $Q_S(2m, \pi, r) \in \overline{\mathbb{Q}}^\times$ by Klingen [15] when n is odd and by Sturm [21, Theorem 1] when n is even. Thus, our choice of f_s and Φ implies that

$$\frac{P_S(m, f, \tau)}{P_S\left(m + \frac{1}{2}, f, \rho\right)} \in \overline{\mathbb{Q}}. \tag{3.4}$$

As we remarked, our algebraicity result in the case of $n = 2, 3$ is a theorem by Shimura [20, Theorem 3]. Hence, by the induction on n, our required algebraicity result follows from (3.4). □

As we remarked in Remark 2, when $n = 4$, Theorem 3.1 gives an algebraicity result for Rankin triple L-functions for GL_2. Since the local Langlands conjecture holds for GL_2, our assertion is for the complete L-function.

Corollary 3.3. *For $i = 1, 2, 3$, let $f_i \in S_{k_i}^{\mathrm{new}}(\Gamma_0(N_i), \varepsilon_i)$ be a Hecke eigenform. Let π_i denote the irreducible unitary cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ associated to f_i. We suppose that:*

1. *$k_2 + k_3$ is even,*
2. *$k_1 > k_2 + k_3 + 4$,*
3. *$\varepsilon_2 \varepsilon_3 = 1$,*
4. *there exists a definite quaternion division algebra D over $\mathbb{Q}$ such that both π_2 and π_3 have the Jacquet-Langlands transfer to $D^\times(\mathbb{A}_\mathbb{Q})$.*

Then for an integer m satisfying

$$\frac{k_1 + k_2 + k_3 + 2}{2} < m \leq k_1 - 1,$$

we have

$$P(m, f_1, f_2, f_3) := \frac{L(m, f_1 \otimes f_2 \otimes f_3)}{\pi^{2(2m-k_1-k_2-k_3+3)} J(f_1)^2} \in \overline{\mathbb{Q}}$$

and

$$P(m, f_1, f_2, f_3)^\sigma = P(m, f_1^\sigma, f_2^\sigma, f_3^\sigma) \quad \textit{for any } \sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right).$$

In particular,

$$P(m, f_1, f_2, f_3) \in \mathbb{Q}(f_1, f_2, f_3).$$

Here we normalize $L(s, f_1 \otimes f_2 \otimes f_3)$ so that it has a functional equation with respect to $s \mapsto k_1 + k_2 + k_3 - 2 - s$, and $\mathbb{Q}(f_1, f_2, f_3)$ is the number field generated by the Fourier coefficients of f_1, f_2 and f_3 over $\mathbb{Q}$.

Before proceeding with a proof of this corollary, we give a remark on above assumptions. The first assumption follows from the third one, but this assumption is important for clarifying our situation, and thus we put it separately.

Proof of Corollary 3.3. From the fourth assumption, there is a Jacquet-Langlands transfer π_i^D of π_i (i=2, 3) to $D^\times(\mathbb{A}_\mathbb{Q})$. Then the third assumption implies that $\pi_2^D \otimes \pi_3^D$ gives an automorphic representation of $\mathrm{SO}(D, \mathbb{A}_\mathbb{Q})$, which is possibly reducible. Let τ_D be one of irreducible constituents of this automorphic representation of $\mathrm{SO}(D, \mathbb{A}_\mathbb{Q})$. Then we know that

$$L(s, \pi_{1,v} \otimes \tau_{D,v}) = L(s, \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v})$$

for every place v of $\mathbb{Q}$. Further, we note that the archimedean component of τ_D is the representation of $\mathrm{SO}(D, \mathbb{R})$ with the highest weight $\left(\frac{k_2+k_3}{2} - 2, \frac{|k_2-k_3|}{2}\right)$. Thus, because of the second assumption, we may apply Theorem 3.1 to π_1 and τ_D. Indeed, taking the normalization of L-functions into account, we have for some finite set of places S of $\mathbb{Q}$,

$$\frac{L_S(m, f_1 \otimes f_2 \otimes f_3)}{\pi^{2(2m-k_1-k_2-k_3+3)} J(f_1)^2} \in \mathbb{Q}(f_1, f_2, f_3)$$

with the integer m such that $\frac{k_1+k_2+k_3+2}{2} < m \leq k_1 - 1$. Furthermore, for every finite place $v \in S$, we have

$$L_v(m, f_1 \otimes f_2 \otimes f_2) \in \overline{\mathbb{Q}}$$

and

$$L_v(m, f_1 \otimes f_2 \otimes f_2)^\sigma = L_v(m, f_1^\sigma \otimes f_2^\sigma \otimes f_2^\sigma)$$

for every $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ (see [10, (4.9.14.1)] and a proof of [7, Therem 2]). Here, this local L-factor is the local L-factor attached to the local Langlands parameter of $\pi_{i,v}$. Then augmenting $L_v(m, f_1 \otimes f_2 \otimes f_2)$ to $L_S(m, f_1 \otimes f_2 \otimes f_3)$ formally, we obtain our required result. □

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